Almost-Periodic Functions in Banach Spaces

By Luigi Amerio

1. *Definition of almost-periodic function. Elementary properties*

The general theory of almost-periodic (a.p.) functions with complex values, created by Harald Bohr in his two classical papers published in Acta Mathematica in 1925 and 1926 [1], has been greatly developed by Weil, De La Vallée-Poussin, Bochner, Stepanov, Wiener, Bogoliubov, Levitan.

Fundamental results, in the theory of a.p. linear ordinary differential equations, are expressed by the theorems of Bohr-Neugebauer and of Favard [2].

Bohr's theory was then, in a particular case, extended by Muckenhaupt [3] and, subsequently, by Bochner [4] and by Bochner and von Neumann [5] to very general abstract spaces.

The extension to Banach spaces has, in particular, revealed itself of great interest, in view of the fundamental importance of these spaces in theory and applications.

Let X be a Banach space; if $x\epsilon X$, we shall indicate by $\|x\|$, or by $\|x\|_X$, the corresponding norm.

Let J be the interval $-\infty<t<+\infty$ and

$$x=f(t) \tag{1.1}$$

a *continuous function* (in the *strong sense*), defined on J and with values in X.

When t varies in J the point $x=f(t)$ describes, in the X space, a set which is called the range of the function $f(t)$, indicated by R_f.

A set $E\subset J$ is said to be *relatively dense* (r.d.) if there exists a number $l>0$ *(inclusion length)* such that every interval $[a, a+l]$ contains at least one point of E.

We shall now say that *the function $f(t)$ is almost-periodic (a.p.) if to every $\varepsilon>0$ there corresponds an r.d. set $\{\tau\}_\varepsilon$ such that*

$$\sup_J \|f(t+\tau)-f(t)\| \le \varepsilon \qquad \forall\tau\epsilon\{\tau\}_\varepsilon. \tag{1.2}$$

Each $\tau\epsilon\{\tau\}_\varepsilon$ is called an *ε-almost period* of $f(t)$; to the set $\{\tau\}_\varepsilon$ therefore corresponds an *inclusion length* l_ε and it is clear that, when $\varepsilon\to 0$, the set $\{\tau\}_\varepsilon$ becomes rarified, while (in general) $l_\varepsilon\to+\infty$.

The above definition was given by Bochner and is an obvious extension of the definition adopted by Bohr for his theory of a.p. functions. It is, undoubtedly, in itself a very significant definition: its real depth can actually be understood only "a posteriori", from the beauty of the theory constructed on it and the importance of its applications.

The theory of a.p. functions with values in a Banach space is, in the way it is treated by Bochner, similar to Bohr's theory of numerical a.p. functions: new developments arise, as is natural, in connection with questions on compactness and boundedness. These questions (which have been particularly studied in Italy) are of notable interest in the integration of a.p. functions and, more generally, in the integration of abstract a.p. partial differential equations [6].

Let us now recall the first properties of a.p. functions, which can be easily deduced from their definition.

We add that when we say that $f(t)$ is uniformly continuous, or bounded, or that the sequence $\{f_n(t)\}$ converges uniformly etc., we always mean that this occurs on the whole interval J.

I *$f(t)$ a.p. $\Rightarrow f(t)$ uniformly continuous (u.c.).*

II *$f(t)$ a.p. $\Rightarrow R_f$ relatively compact (r.c.)* (that is the closure $\bar{R}_f$ is compact).

III *$f_n(t)$ a.p. ($n = 1, 2, \ldots$), $f_n(t) \to f(t)$ uniformly $\Rightarrow f(t)$ a.p.*

IV *$f(t)$ a.p., $f'(t)$ uniformly continuous $\Rightarrow f'(t)$ a.p.*

V *$x = f(t)$ X-a.p., $y = g(x)$ with values in Y* (Banach) *and continuous on $\bar{R}_f \Rightarrow g(f(t))$ Y-a.p.*

In particular:

$$f(t) \text{ a.p.}, k>0 \Rightarrow \|f(t)\|^k \text{ a.p.}$$

2. *Bochner's criterion*

The class of a.p. functions has been characterized by Bochner by means of a compactness criterion, which plays an essential role in the theory and in applications. The starting point consists in considering, together with a given function $f(t)$, the set of its translates $\{f(t+s)\}$ and its closure $\overline{\{f(t+s)\}}$ with respect to uniform convergence. We have then:

VI *Let $f(t)$ be continuous, from J to X. A necessary and sufficient condition for $f(t)$ to be a.p. is that from every sequence $\{s_n\}$ it may be possible to extract a subsequence $\{l_n\}$ such that the sequence $\{f(t+l_n)\}$ be uniformly convergent.*

A very important consequence of Bochner's criterion is that *the sum $f(t) + g(t)$ of two X - a.p. functions is X - a.p.; the product $\varphi(t)f(t)$ of $f(t)$, X - a.p., by a numerical a.p. function $\varphi(t)$, is a.p.* It follows, in particular, the almost-periodicity of all *trigonometric polynomials:*

$$P(t) = \sum_{1}^{n} a_k e^{i\lambda_k t} \quad (a_k \in X, \lambda_k \in J).$$

Observation. Let $x = f(t) \in L^p_{\text{loc}}(J; X)$, with $1 \leq p < +\infty$: assume in other words, that $\int_\Delta \|f(t+\eta)\|^p d\eta < +\infty \quad \forall t \in J$, where $\Delta = [0,1]$.

The function $f(t)$ is said to be *a.p. in the sense of Stepanov if to every* $\varepsilon > 0$ *there corresponds an r.d. set* $\{\tau\}_\varepsilon$ *such that*

$$\sup_J \{\int_\Delta \|f(t+\tau+\eta) - f(t+\eta)\|^p d\eta\} \leq \varepsilon \quad \forall \tau \in \{\tau\}_\varepsilon. \tag{1.3}$$

As has been observed by Bochner, *the almost-periodicity in the sense of Stepanov can be reduced to that in the sense of Bohr (for vector valued functions).* Consider, in fact, the Banach space $L^p(\Delta; X)$ and define, $\forall t \in J$, the vector $\tilde{f}(t) = \{f(t+\eta)\} \in L^p(\Delta; X)$. We have then

$$\{\int_\Delta \|f(t+\tau+\eta) - f(t+\eta)\|^p d\eta\}^{1/p} = \|\tilde{f}(t+\tau) - \tilde{f}(t)\|_{L^p(\Delta; X)}$$

and the thesis follows from (1.3).

3. *Harmonic analysis of almost-periodic functions*

The harmonic analysis of a.p. functions extends to these the theory of Fourier expansions of periodic functions. The following statements hold:

VII *(approximation theorem). If $f(t)$ is a.p. there exists, $\forall \varepsilon > 0$, a trigonometric polynomial $P_\varepsilon(t)$ such that*

$$\sup_J \|f(t) - P_\varepsilon(t)\| \leq \varepsilon.$$

VIII *(theorem of the mean). If $f(t)$ is a.p. there exists the mean value*

$$M(f(t)) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, dt$$

It follows that the function of λ

$$a(\lambda; f) = M(f(t)e^{-i\lambda t})$$

is defined on J; $a(\lambda; f)$ takes its values in X, as does $f(t)$: we shall call this function the *Bohr transform* of the a.p. function $f(t)$.

It can be seen, by VII, that $a(\lambda; f) = 0$ *on the whole of J, with the exclusion, at most, of a sequence* $\{\lambda_n\}$.

The values λ_n for which $a_n = a(\lambda_n; f) \neq 0$ are called the *characteristic* exponents of $f(t)$. The vectors a_n are the *Fourier coefficients of* $f(t)$, to which we can associate the *Fourier series*

$$f(t) \sim \sum_{1\ n}^{\infty} a_n e^{i\lambda_n t}.$$

IX *(uniqueness theorem)*

$$f(t) \text{ and } g(t) \ X\text{-a.p.}, \ a(\lambda; f) \equiv a(\lambda; g) \Rightarrow f(t) \equiv g(t).$$

The *correspondence* between *almost-periodic functions* and their *Bohr transforms* is therefore *one-to-one*. A property of the transform $a(\lambda; f)$ is given by the following proposition:

X $a(\lambda'; f) = 0 \Rightarrow \lim_{\lambda\to\lambda'} a(\lambda; f) = 0$, *that is the Bohr transform is continuous at all points in which it vanishes. Furthermore:*

$$\lim_{\lambda\to\infty} a(\lambda, f) = 0 \quad , \quad \lim_{n\to\infty} a_n = 0,$$

and, for Hilbert spaces:

$$M(\|f(t)\|^2) = \sum_{1\ n}^{\infty} \|a_n\|^2 \qquad \text{(Parseval's equality).}$$

We recall moreover that *Bochner's approximation polynomial* can be constructed also in the abstract case.

4. *Weakly almost-periodic functions*

Given the Banach space X, we shall call X^* its dual space (a Banach space too) constituted by the linear functionals continuous on X. If $x \in X$, $x^* \in X^*$, we shall indicate by $<x^*, x>$ the complex value that, through the functional x^*, corresponds to x, and by $\|x^*\|$ the norm of x^*.

We shall say that $f(t)$, with values in X, is *weakly almost periodic (w.a.p.) if,* $\forall x^* \in X^*$, *the numerical function*

$$<x^*, f(t)>$$

is a.p. [7].

As may be seen, the definition given here has, with respect to that of an *a.p.* function, the same relation as the definition of weakly continuous function has with respect to that of continuous function.

Its interest is particularly connected with statement XIV below. A different definition of weak almost periodicity is due to Eberlein [8]: the *w.a.p.* functions in the sense of Eberlein possess notable properties, particularly in relation to ergodic theorems.

It is clear (as $<x^*, x> \leq \|x^*\| \|x\|$) that $f(t)$ *a.p.* $\Rightarrow f(t)$ *w.a.p.* In order to indicate that $\{x_n\}$ is a sequence *converging weakly* to x (i.e. $<x^*, x_n> \rightarrow <x^*, x>$, $\forall x^* \in X^*$) we shall make use of all the following notations:

$$x_n \xrightarrow{*} x, \text{ or } \lim_{n\to\infty}{}^* x_n = x,$$

and x is called the weak limit (which, if it exists, is also unique) of the sequence $\{x_n\}$. Let us remember that, in an arbitrary Banach space, a sequence $\{x_n\}$ can be *scalarly convergent* (i.e. $\lim <x^*, x_n>$ exists and is finite $\forall x^* \in X^*$) without necessarily being weakly convergent, that is without there being an x which is its weak limit. If this circumstance is not present (i.e. if scalar convergence implies weak convergence) the space X is said to be *semicomplete* (reflexive, and, in particular, Hilbert spaces are semicomplete).

Let us now indicate some properties of *w.a.p.* functions.

XI *$f(t)$ w.a.p. $\Rightarrow R_f$ bounded and separable.*

When necessary, we can therefore assume that X is separable.

XII *$f_n(t)$ w.a.p. ($n = 1, 2, \ldots$), $f_n(t) \xrightarrow{*} f(t)$ uniformly $\Rightarrow f(t)$ w.a.p.* ($f_n(t) \xrightarrow{*} f(t)$ uniformly means that, $\forall x^* \in X^*$, $<x^*, f_n(t)> \rightarrow <x^*, f(t)>$ *uniformly).*

XIII *Let X be semicomplete and $f(t)$ weakly continuous. Then $f(t)$ w.a.p. $\Leftrightarrow \forall \{s_n\}$ there exists a subsequence $\{s'_n\}$ such that $\{f(t + s'_n)\}$ is uniformly weakly convergent.*

This proposition extends Bochner's criterion to *w.a.p.* functions (though with a restrictive hypothesis on the nature of the space X).

As we have already observed, $f(t)$ *a.p.* $\Rightarrow f(t)$ *w.a.p.* It is important to note that the property that has to be added to weak almost-periodicity to obtain almost-periodicity is one of *compactness*. The following theorem can, in fact, be proved.

XIV *$f(t)$ w.a.p. and R_f r.c. $\Rightarrow f(t)$ a.p.*

5. *Integration of almost-periodic functions*

If $f(t)$ is an *a.p.* function with values in a Banach space X, we will write, in what follows,

$$F(t) = \int_0^t f(\eta)\, d\eta. \tag{5.1}$$

The problem of the integration of *a.p.* functions in Banach spaces is of notable interest, also because it serves, so to say, as a *model* for classifying Banach spaces in relation to the theory of abstract *a.p.* equations.

If X is Euclidean, then Bohr's theorem holds: $F(t)$ bounded $\Rightarrow$ $F(t)$ a.p.

For the *general case (X arbitrary Banach space), the almost-periodicity of $F(t)$ has been proved by Bochner under the hypothesis that R_F is r.c.*

This condition is obviously much more restrictive than that of boundedness; it can not however be substituted in the general case by the latter, as can be shown in the following example (Amerio, [9]).

Consider, in fact, the space l^∞ of bounded sequences of complex numbers: $x = \{\xi_n\}$, with $\|x\| = \operatorname{Sup}_n |\xi_n|$. The function $f(t) = \{n^{-1} \cos(t/n)\}$ is *a.p.* and has the integral $F(t) = \{\sin(t/n)\}$, which is *bounded* ($\|F(t)\| \leq 1$) and *weakly a.p.* (see a) below), but *not a.p.*

One can prove nevertheless [9] that *Bohr's enunciation remains unaltered if the space X is uniformly convex* (it holds therefore in *Hilbert spaces*, in l^p and L^p, with $1 < p < +\infty$).

Let us prove now the following theorems.

XV (Bochner) *X arbitrary, $f(t)$ a.p., R_F r.c. $\Rightarrow$ $F(t)$ a.p.*
XVI (Amerio) *X uniformly convex, $f(t)$ a.p., $F(t)$ bounded $\Rightarrow$ $F(t)$ a.p.*

a) Proof of the theorem XV. As R_F is r.c., $F(t)$ is bounded:

$$\operatorname{Sup}_J \|F(t)\| = M < +\infty. \tag{5.2}$$

Furthermore, $\forall x^* \in X^*$,

$$|\langle x^*, F(t)\rangle| = |\langle x^*, \int_0^t f(\eta)\, d\eta\rangle| = |\int_0^t \langle x^*, f(\eta)\rangle d\eta| \leq \|x^*\|\, M.$$

As $\langle x^*, f(t)\rangle$ is *a.p.*, from Bohr's theorem it follows that $\langle x^*, F(t)\rangle$ is *a.p.*; $F(t)$ is therefore *w.a.p.*

R_F has been supposed r.c.; our thesis follows then from theorem XIV.

b) Proof of theorem XVI. We have already proved in a) (utilizing only the boundedness of $F(t)$) that $F(t)$ is *w.a.p.* It is therefore sufficient, making use of the properties of uniformly convex spaces, to prove that R_F is r.c.

We first of all remember that a space X is called a uniformly convex (or Clarkson) space if in the interval $0<\sigma\leq 2$ there exists a function $\omega(\sigma)$, with $0<\omega(\sigma)\leq 1$, such that

$$\|x\|, \|y\| \leqslant 1 \text{ and } \|x-y\| \geqslant \sigma \Rightarrow \left\|\frac{x+y}{2}\right\| \leqslant 1-\omega(\sigma). \tag{5.3}$$

Now we observe that from (5.3) it follows for any x and y:

$$\|x-y\| \geqslant \sigma \max \{ \|x\|, \|y\|\} \Rightarrow \left\|\frac{x+y}{2}\right\| \leqslant (1-\omega(\sigma)) \max \{\|x\|, \|y\|\}. \tag{5.4}$$

Let us assume that the range R_F is not r.c. There exist then a constant $\delta>0$ and a sequence $\{s_n\}$ such that

$$\|F(s_j) - F(s_k)\| \geqslant \delta \qquad (j\neq k). \tag{5.5}$$

We can suppose that $\{s_n\}$ is *regular* with respect to $f(t)$ and $F(t)$, that is

$$\lim_{n\to\infty} f(t+s_n) = f_s(t), \qquad \lim_{n\to\infty}{}^{*} F(t+s_n) = F_s(t) \tag{5.6}$$

uniformly. The last relation follows from Bochner's criterion (theorem XIII), noting that the space X is semicomplete (being reflexive).

It also holds that

$$F(t+s_j) = F(s_j) + \int_0^t f(\eta+s_j)\, d\eta$$

and, consequently, for $j\neq k$,

$$\|F(t+s_j) - F(t+s_k)\| \geqslant \|F(s_j) - F(s_k)\| - \left\|\int_0^t (f(\eta+s_j) - f(\eta+s_k))\, d\eta\right\|.$$

If we fix $t\in J$, we will have, by (5.5) and the former part of (5.6),

$$\|F(t+s_j) - F(t+s_k)\| \geqslant \frac{\delta}{2} \text{ for } j > k \geqslant n_t.$$

Therefore, by (5.2),

$$\|F(t+s_j) - F(t+s_k)\| \geqslant \frac{\delta}{2M} \max \{\|F(t+s_j)\|, \|F(t+s_k)\|\}$$

and, by (5.4),

$$\frac{1}{2}\| F(t+s_j) + F(t+s_k)\| \leqslant (1 - \omega(\frac{\delta}{2M})) \max \{\|F(t+s_j)\|, \|F(t+s_k)\|\} \leqslant$$

$$\leqslant (1-\omega(\frac{\delta}{2M}))\, M.$$

From the latter part of (5.6) it then follows

$$\|F_s(t)\| \leqslant (1-\omega(\frac{\delta}{2M}))\, M$$

and, consequently,

$$\operatorname*{Sup}_{t\in J} \|F_s(t)\| \leqslant (1-\omega(\frac{\delta}{2M}))\, M. \qquad (5.7)$$

Relation (5.7) is absurd; from the latter part of (5.6) follows in fact, the weak convergence being uniform,

$$\lim_{n\to\infty}{}^{*} \quad F_s(t-s_n) = F(t)$$

and therefore

$$\|F(t)\| \leqslant \liminf_{n\to\infty} \|F_s(t-s_n)\| \leqslant (1-\omega(\frac{\delta}{2M}))\, M,$$

which contradicts (5.2).

It is of interest to note that the previously given example is, in a certain sense, the only possible. Both functions $f(t)$ and $F(t)$ belong in fact to the subspace c_0 of l^∞, of numerical sequences which converge to 0. The analysis of Banach spaces X which do not contain c_0 is due to Pelczynski [10], and the important role of these spaces in the problem of integration was indicated by Kadets [11]. The following theorem in fact holds:

XVII (Kadets) Assume *$f(t)$ a.p., $F(t)$ bounded. Then $F(t)$ is a.p. if and only if the space X does not contain c_0.*

Observation. As we have observed in §1, the above considerations are essential in the study of some typical equations, linear or non linear, of mathematical and theoretical physics; in particular [12]: the wave equation, Schrödinger's equation with time-dependent operator, and, in the non linear field, the wave equation with non linear dissipative term and the Navier-Stokes equation (assuming, in all cases, the presence of an *a.p.* forcing term $f(t)$, and setting the problems in Hilbert or uniformly convex spaces).

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Politecnico di Milano
Dipartimento di Matematica
20133 Milano, Piazza Leonardo da Vinci 32
Italy